

# A CURIOUS IDENTITY AND THE VOLUME OF THE ROOT SPHERICAL SIMPLEX.

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with an appendix by

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*A Guido Zappa per i suoi 90 anni*

**ABSTRACT.** We show a curious identity on root systems which gives the evaluation of the volume of the spherical simplexes cut by the cone generated by simple roots. In the appendix John Stembridge gives a conceptual proof of our identity

## 1. INTRODUCTION

In this note we shall consider a finite root system  $R$  spanning an euclidean space  $E$  of dimension  $\ell$  (for all the facts about root systems which we are going to use in this note we refer to [1]).  $\ell$  is called the rank of  $R$ . We shall choose once and for all a set of positive roots  $R^+$  and in  $R^+$  the set of simple roots  $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ . We shall also denote by  $W$  the Weyl group of  $R$  i.e. the finite group generated by the reflections with respect to the hyperplanes orthogonal to the roots in  $R$ . Given such a root  $\alpha \in R$ , we shall denote by  $s_\alpha \in W$  the reflection with respect to the hyperplane orthogonal to  $\alpha$ . Set  $s_i = s_{\alpha_i}$  for each  $i = 1, \dots, \ell$  and call  $S = \{s_1, \dots, s_\ell\}$  the set of simple reflections. One know that  $S$  generates  $W$  and that the pair  $(W, S)$  is a Coxeter group.

We know that the ring of regular functions on  $E$ , invariant under the action of  $W$ , is a polynomial ring generated by homogenous elements of degrees  $d_1 \leq d_2 \leq \dots \leq d_\ell$ . The  $d_i$ 's are called the degrees. We shall also consider the sequence of exponents,  $d_1 - 1, d_2 - 1, \dots, d_\ell - 1$ . Recall that  $\prod_i d_i = |W|$ .

In  $E$  we have the *affine arrangement* of the hyperplanes orthogonal to the roots and their translates under the weight lattice  $\Lambda$ , a locally finite configuration invariant under the affine Weyl group  $\hat{W}$ .  $\hat{W}$  is the semidirect product of  $W$  and of the lattice  $Q$  spanned by the roots, thought of as translation operators.

$\hat{W}$  is itself a Coxeter group. In the case in which  $E$  is irreducible, its Coxeter generators are given by the reflections  $\{s_0, s_1, \dots, s_\ell\}$ , where the for

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$i \geq 1$  the  $s_i$ 's are the simple generators of  $W$  and

$$s_0(v) = s_\theta(v) + \theta$$

$\theta$  being the longest root. One knows that, for each  $0 \leq i \leq \ell$ , the subgroup  $W_i$  of  $\hat{W}$  generated by the reflections  $(s_0, \dots, \check{s}_i, \dots, s_\ell)$  is finite, and it is the Weyl group of a root system  $R^{(i)}$  which will be discussed presently. Hence we can consider the degrees  $d_1^{(i)} \leq d_2^{(i)} \leq \dots \leq d_\ell^{(i)}$ . Our main result is the identity (Theorem 1.2)

$$\sum_{i=0}^{\ell} \frac{(d_1^{(i)} - 1)(d_2^{(i)} - 1) \cdots (d_\ell^{(i)} - 1)}{d_1^{(i)} d_2^{(i)} \cdots d_\ell^{(i)}} = 1.$$

The proof is a case by case computation using the classification of irreducible root systems. It is quite desirable to give a more conceptual deduction of our identity.

In the last section we show, following a suggestion of Vinberg, that our identity implies the following geometric identity. Take the unit sphere  $S(E)$  in  $E$  and consider the spherical simplex  $S(E) = C(\Delta) \cap S(E)$ ,  $C(\Delta)$  being the cone of positive linear combinations of the simple roots. Then

$$\frac{\text{Vol } S(\Delta)}{\text{Vol } S(E)} = \frac{(d_1 - 1)(d_2 - 1) \cdots (d_\ell - 1)}{d_1 d_2 \cdots d_\ell}.$$

We have discovered this identity while trying to understand the following fact.

Consider the complex space  $V = E \otimes_{\mathbb{R}} \mathbb{C}$ , and take the algebraic torus  $T = V/Q$ . For any root  $\alpha \in R$  the linear form  $\check{\alpha}$  defined by

$$\check{\alpha}(v) = 2 \frac{(\alpha, v)}{(\alpha, \alpha)}$$

takes integer values on  $Q$ , hence we get the character  $e^{2\pi\sqrt{-1}\check{\alpha}}$  of  $T$ .

Denote its kernel by  $D_\alpha$ . In our work on toric arrangements (see [2], [5], [6], [7], [8]) we have shown that the Euler characteristic of the open set  $\mathcal{A} := T - \cup_{\alpha \in R^+} D_\alpha$  equals  $(-1)^\ell |W|$ . The only proof we know of this fact is via a combinatorial topological construction of Salvetti [4] [3]. The above identity has been the result of an attempt to give a direct computation of this Euler characteristic.

**1.1. The main identity.** We are interested in the numbers

$$\nu(R) = \prod_{i=1}^{\ell} \frac{d_i - 1}{d_i}.$$

The following table gives  $\nu(R)$  in the case of irreducible root systems

$$\begin{aligned} A_n, n \geq 1, \quad \nu_{A_n} &= \frac{1}{n+1} \\ B_n \text{ and } C_n, n \geq 2, \quad \nu_{B_n} &= \frac{1}{4^n} \binom{2n}{n} \end{aligned}$$

$$\begin{aligned}
D_n, n \geq 4, \quad \nu_{D_n} &= \frac{n-1}{4^{n-1}n} \binom{2(n-1)}{n-1} \\
\nu_{G_2} &= \frac{5}{12} \\
\nu_{F_4} &= \frac{385}{1152} \\
\nu_{E_6} &= \frac{77}{324} \\
\nu_{E_7} &= \frac{2431}{9216} \\
\nu_{E_8} &= \frac{30808063}{99532800}
\end{aligned}$$

Notice that if  $R$  is reducible, i.e.  $R = R_1 \cup R_2$  with  $R_1 \perp R_2$ , then clearly

$$\nu(R) = \nu(R_1)\nu(R_2).$$

We normalize the scalar product so that the short roots have length  $\sqrt{2}$  and we denote by  $D$  the Dynkin diagram of  $R$ .

From now on we assume that  $R$  is irreducible and we denote by  $\hat{D}$  the extended Dynkin diagram. Set  $\alpha_0 = -\theta$ , with  $\theta$  the highest root. We have a bijection between the set  $\hat{\Delta} = \{\alpha_0, \dots, \alpha_\ell\}$  and the nodes of  $\hat{D}$ . For every  $i = 0, \dots, \ell$  the diagram  $D_i$  obtained from  $\hat{D}$  removing the node corresponding to the root  $\alpha_i$  (and all the edges having that node as one of the vertices) is of finite type. So we can consider the corresponding root system  $R^{(i)}$  consisting of all roots in  $R$  which are integral linear combinations of the roots  $\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_\ell$  and the corresponding number  $\nu(R^{(i)})$ .

During the proof of our result we shall need the following well known

**Lemma 1.1.** *The following identities hold,*

$$(1) \quad 4^n = \sum_{h=0}^n \binom{2h}{h} \binom{2(n-h)}{n-h}, \quad n \geq 0$$

Furthermore, when  $n \geq 2$  we have:

$$(2) \quad 4^{n-1} = \frac{1}{2} \binom{2n}{n} + \sum_{h=2}^n \frac{h-1}{h} \binom{2(h-1)}{h-1} \binom{2(n-h)}{n-h}$$

$$(3) \quad 4^{n-2} = \frac{n-1}{4n} \binom{2(n-1)}{n-1} + \sum_{h=2}^{n-2} \frac{(h-1)(n-h-1)}{h(n-h)} \binom{2(h-1)}{h-1} \binom{2(n-1-h)}{n-1-h}.$$

*Proof.* The first identity follows immediately from the following power series expansion

$$(4) \quad \frac{1}{\sqrt{1-4t}} = \sum_{n \geq 0} \binom{2n}{n} t^n.$$

To see this notice that setting

$$f(t) = \frac{1}{\sqrt{1-4t}}$$

we have

$$\frac{df(t)}{dt} = 2f(t)^3$$

from which we deduce that

$$(1-4t)\frac{df(t)}{dt} = 2f(t).$$

Writing  $f(t) = \sum_{n \geq 0} a_n t^n$  we deduce that

$$\sum_{n \geq 0} n a_n t^{n-1} - 4 \sum_{n \geq 0} n a_n t^n = 2 \sum_{n \geq 0} a_n t^n.$$

Equating coefficients, we get  $a_0 = 1$  and for  $h \geq 0$

$$a_{h+1} = \frac{2(2h+1)}{h+1} a_h.$$

On the other hand if we set  $b_h := \binom{2h}{h}$ , we get  $b_0 = 1$  and

$$b_{h+1} = \frac{2(h+1)!}{(h+1)!(h+1)!} = \frac{2(h+1)(2h+1)}{(h+1)^2} b_h = \frac{2(2h+1)}{h+1} b_h$$

so  $a_n = b_n$  and everything follows.

To see the second identity, notice that using (4) and integrating, we get

$$(5) \quad \frac{1}{2} - \sum_{h \geq 1} \frac{1}{h} \binom{2(h-1)}{h-1} t^h = \frac{1}{2} \sqrt{1-4t}$$

Again using (4) we deduce

$$\begin{aligned} \frac{1}{2} + \sum_{h \geq 2} \frac{h-1}{h} \binom{2(h-1)}{h-1} t^h &= \sum_{h \geq 1} \binom{2(h-1)}{h-1} t^h + \frac{1}{2} - \sum_{h \geq 1} \frac{1}{h} \binom{2(h-1)}{h-1} t^h = \\ (6) \quad &= \frac{1-2t}{2\sqrt{1-4t}} \end{aligned}$$

This together with (4) implies that

$$\frac{1}{2} \binom{2n}{n} + \sum_{h=2}^n \frac{h-1}{h} \binom{2(h-1)}{h-1} \binom{2(n-h)}{n-h}$$

is the coefficient of  $t^n$  in the power series expansion of

$$\frac{1-2t}{2\sqrt{1-4t}} \frac{1}{\sqrt{1-4t}} = \frac{1}{2} + \frac{t}{1-4t}$$

since  $n \geq 2$  the claim follows. To see the last identity, let us remark that its left handside is the coefficient of  $t^n$  in the power series expansion of the function

$$\left(\frac{1-2t}{2\sqrt{1-4t}}\right)^2 = \frac{1+4t^2-4t}{1-4t}.$$

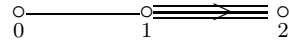
From this everything follows.  $\square$

**Theorem 1.2.**  $\sum_{i=0}^{\ell} \nu(R^{(i)}) = 1$

*Proof.* The proof is by a case by case computation.

Let us deal first with the exceptional cases. In order to make the computation transparent it is more convenient to multiply our sum by  $|W|$ , the order of the Weyl group

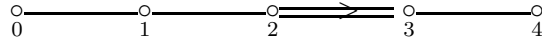
Case  $G_2$ . In this case  $|W| = 12$ . By looking at the extended Dynkin diagram



we get that

$$|W| \sum_{i=0}^2 \nu(R^{(i)}) = 12(\nu(G_2) + \nu(A_2) + \nu(A_1 \times A_1)) = 5 + 3 + 4 = 12$$

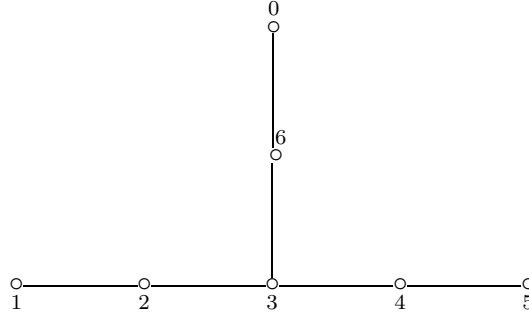
Case  $F_4$ . The order of the Weyl group is 1152. By looking at the extended Dynkin diagram



we get that

$$\begin{aligned} |W| \sum_{i=0}^4 \nu(R^{(i)}) &= |W|(\nu(F_4) + \nu(A_1 \times C_3) + \nu(A_2 \times A_2) + \nu(A_3 \times A_1) + \nu(B_4)) = \\ &= 385 + 180 + 128 + 144 + 315 = 1152. \end{aligned}$$

Case  $E_6$ . In this case  $|W| = 51840$ . By looking at the extended Dynkin diagram

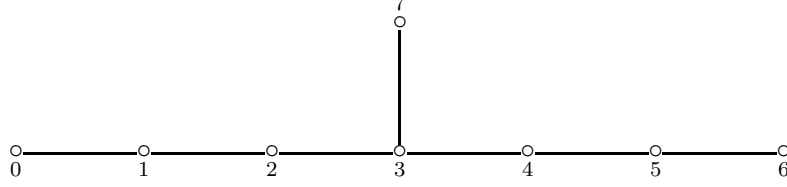


we get that

$$|W| \sum_{i=0}^6 \nu(R^{(i)}) = |W|(3\nu(E_6) + \nu(A_2 \times A_2 \times A_2) + 3\nu(A_1 \times A_5)) =$$

$$= 36960 + 1920 + 12960 = 51840$$

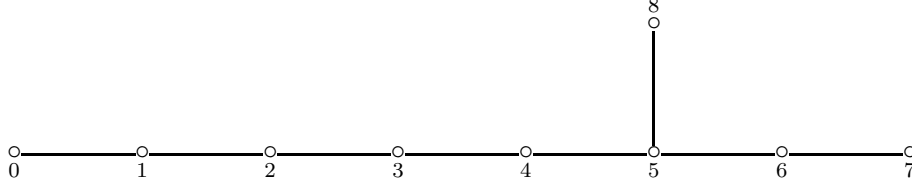
Case  $E_7$ . In this case  $|W| = 2903040$ . By looking at the extended Dynkin diagram



we get that

$$\begin{aligned} |W| \sum_{i=0}^7 \nu(R^{(i)}) &= |W| (2\nu(E_7) + 2\nu(A_1 \times D_6) + 2\nu(A_2 \times A_5) + \nu(A_3 \times A_3 \times A_1) + \nu(A_7)) = \\ &= 1531530 + 595350 + 322560 + 90720 + 362880 = 2903040 \end{aligned}$$

Case  $E_8$ . In this case  $|W| = 696729600$ . By looking at the extended Dynkin diagram



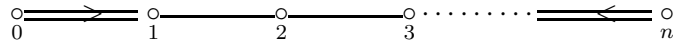
we get that

$$\begin{aligned} |W| \sum_{i=0}^8 \nu(R^{(i)}) &= |W| (\nu(E_8) + \nu(A_1 \times E_7) + \nu(A_2 \times E_6) + \nu(A_3 \times D_5) + \nu(A_4 \times A_4) + \\ &\quad + \nu(A_5 \times A_2 \times A_1) + \nu(A_1 \times A_7) + \nu(D_8) + \nu(A_8)) = \\ &= 215656441 + 91891800 + 55193600 + 38102400 + 27869184 + \\ &\quad + 19353600 + 43545600 + 127702575 + 77414400 = 696729600 \end{aligned}$$

Case  $A_n$ . In this case each  $R^{(i)}$  is of type  $A_n$ . It follows that

$$\sum_{i=0}^n \nu(R^{(i)}) = (n+1)\nu_{A_n} = (n+1)\frac{1}{n+1} = 1$$

Case  $C_n$ . The extended Dynkin diagram is

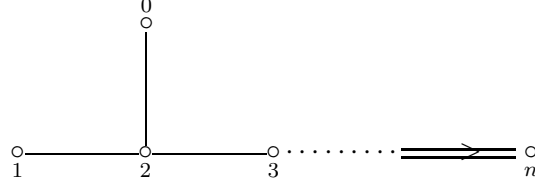


we get that, denoting by  $C_0$  the trivial root system and setting  $C_1 = A_1$ ,

$$\sum_{i=0}^n \nu(R^{(i)}) = \sum_{h=0}^n \nu(C_h \times C_{n-h}) = \sum_{h=0}^n \frac{1}{4^{h+n-h}} \binom{2h}{h} \binom{2(n-h)}{n-h} = 1$$

by Lemma 1.1, part (1).

Case  $B_n$ . The extended Dynkin diagram is

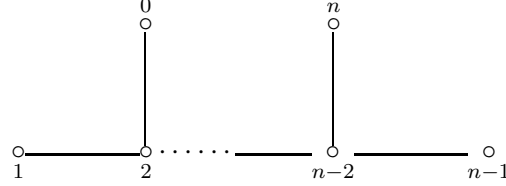


we get that, denoting by  $B_0$  the trivial root system, setting  $C_1 = A_1$ ,  $D_2 = A_1 \times A_1$  and  $D_3 = A_3$ ,

$$\begin{aligned} \sum_{i=0}^n \nu(R^{(i)}) &= 2\nu(B_n) + \sum_{h=2}^n \nu(D_h \times B_{n-h}) = \\ &= \frac{2}{4^n} \binom{2n}{n} + \sum_{h=2}^n \frac{h-1}{h} \binom{2(h-1)}{h-1} \binom{2(n-h)}{n-h} = \\ &= \frac{1}{4^{n-1}} \left( \frac{1}{2} \binom{2n}{n} + \sum_{h=2}^n \frac{h-1}{h} \binom{2(h-1)}{h-1} \binom{2(n-h)}{n-h} \right) = 1 \end{aligned}$$

by Lemma 1.1, part (2).

Case  $D_n$ . The extended Dynkin diagram is



we get that, setting  $D_2 = A_1 \times A_1$  and  $D_3 = A_3$ ,

$$\begin{aligned} \sum_{i=0}^n \nu(R^{(i)}) &= 4\nu(D_n) + \sum_{h=2}^{n-2} \nu(D_h \times D_{n-h}) = \\ &= \frac{1}{4^{n-2}} \left( \frac{n-1}{4n} \binom{2(n-1)}{n-1} + \sum_{h=2}^{n-2} \frac{(h-1)(n-h-1)}{h(n-h)} \binom{2(h-1)}{h-1} \binom{2(n-1-h)}{n-1-h} \right) \end{aligned}$$

which equals 1 by Lemma 1.1, part (3).  $\square$

**1.2. The volume of  $S(\Delta)$ .** Recall that we have introduced the spherical simplex as the intersection of the unit sphere  $S(E)$  in  $E$  with the cone  $C(\Delta)$  of non negative linear combinations of the simple roots  $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$  for the root system  $R$ . Our purpose is to show

**Theorem 1.3.**

$$\frac{\text{Vol } S(\Delta)}{\text{Vol } S(E)} = \nu(R) = \frac{(d_1 - 1)(d_2 - 1) \cdots (d_\ell - 1)}{d_1 d_2 \cdots d_\ell}.$$

*Proof.* For simplicity we normalize in such a way that  $\text{Vol } S(E) = 1$ . We then set  $\text{Vol } S(\Delta) = V(R)$ . If  $R$  is reducible, i.e.  $R = R_1 \cup R_2$  with  $R_1 \perp R_2$ , we have

$$V(R) = V(R_1)V(R_2).$$

Since we also have

$$\nu(R) = \nu(R_1)\nu(R_2),$$

an easy induction implies that we are reduced to show our claim under the assumption that  $R$  is irreducible.

So assume  $R$  irreducible and set  $\alpha_0 = -\theta$ , with  $\theta$  the highest root. Write  $\alpha_0 = \sum_{j=1}^{\ell} n_j \alpha_j$  with  $n_j$  a negative integer for all  $j = 1, \dots, \ell$ .

As in the previous section for every  $i = 0, \dots, \ell$  set  $R^{(i)}$  equal to the root system consisting of all roots in  $R$  which are integral linear combinations of the roots  $\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_{\ell}$  so that in particular  $|R^{(i)}| \leq |R|$ . Recall that the Dynkin diagram of  $R^{(i)}$  is the subdiagram of  $\hat{D}$  obtained by removing the node corresponding to  $\alpha_i$ . The roots  $\Delta^{(i)} = \{\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_{\ell}\}$  are simple roots for  $R^{(i)}$ .

We claim that  $E$  is the union of the cones  $C(\Delta^{(i)})$  whose interior are disjoint. To see this take  $u \in E$ , write  $u = \sum_{h=1}^{\ell} b_h \alpha_h$ . If all  $b_h$  are larger or equal than zero then  $u \in C(\Delta) = C(\Delta^{(0)})$ , otherwise  $b_h < 0$  for at least one index  $1 \leq h \leq \ell$ . Take an index  $i$  for which  $b_i/n_i$  is maximum. Notice that necessarily  $b_i/n_i > 0$ . We can clearly write

$$u = \frac{b_i}{n_i} \alpha_0 + \sum_{h=1, h \neq i}^{\ell} (b_h - \frac{n_h b_i}{n_i}) \alpha_h$$

and all coefficients are non negative.

Now observe that if, for any  $i = 0, \dots, \ell$ , we write  $\alpha_i$  as a linear combination of  $\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_{\ell}$  then all coefficients are negative. We then leave to the reader the easy verification that this implies that the interiors of the cones  $C(\Delta^{(i)})$  are mutually disjoint.

We deduce that

$$(7) \quad \sum_{h=0}^{\ell} V(R^{(h)}) = 1.$$

Now set  $\Gamma = \{i | R^{(i)} = R\}$ .  $\Gamma$  is not empty since  $0 \in \Gamma$ . We can rewrite (7) as

$$|\Gamma|V(R) + \sum_{h \notin \Gamma} V(R^{(h)}) = 1.$$

Similarly by Theorem 1.2 we get

$$|\Gamma|\nu(R) + \sum_{h \notin \Gamma} \nu(R^{(h)}) = 1.$$



Since, by the definition of  $\Gamma$ , for  $h \notin \Gamma$  we have  $|R^{(h)}| < |R|$ , by induction (the case  $A_1$  in which we have 2 roots is trivial) we can assume  $V(R^{(h)}) = \nu(R^{(h)})$ .

We get

$$V(R) = \frac{1}{|\Gamma|} \left(1 - \sum_{h \notin \Gamma} V(R^{(h)})\right) = \frac{1}{|\Gamma|} \left(1 - \sum_{h \notin \Gamma} \nu(R^{(h)})\right) = \nu(R)$$

proving our claim □

#### REFERENCES

- [1] Bourbaki N., *Groupes et algèbres de Lie, Chapitres 4,5,6* Actualités Scientifiques et Industrielles, No. 1337 Hermann, Paris 1968 288 pp.
- [2] De Concini, C.; Procesi, C. *On the geometry of toric arrangements*. Transform. Groups 10 (2005), no. 3-4, 387–422.
- [3] De Concini, C.; Salvetti, M. *Cohomology of Artin groups: Addendum: "The homotopy type of Artin groups"* Math. Res. Lett. 3 (1996), no. 2, 293–297.
- [4] Salvetti, Mario *The homotopy type of Artin groups*. Math. Res. Lett. 1 (1994), no. 5, 565–577.
- [5] A. Henderson, *The symmetric group representation on cohomology of the regular elements of a maximal torus of the special linear group*, preprint <http://www.maths.usyd.edu.au/u/anthonyh/>
- [6] Lehrer, G. I. *The cohomology of the regular semisimple variety*. J. Algebra 199 (1998), no. 2, 666–689.
- [7] Lehrer, G. I. *A toral configuration space and regular semisimple conjugacy classes*. Math. Proc. Cambridge Philos. Soc. 118 (1995), no. 1, 105–113.
- [8] E. Looijenga, *Cohomology of  $\mathcal{M}_3$  and  $\mathcal{M}_3^1$* . Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., 150, Amer. Math. Soc., 1993, 205–228.

## APPENDIX

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In this appendix, we provide an explanation for the “curious identity” (Theorem 1.2) without any case-by-case considerations. The proof is based on two elegant formulas, one due to L. Solomon, the other due to R. Steinberg. Both of these results deserve to be better known.

If  $W$  is a finite group generated by reflections in a real Euclidean space  $E$ , consider the class function on  $W$  defined by

$$\delta_W(q, t)(w) := \frac{\det(1 - qw)}{\det(1 - tw)} \quad (w \in W),$$

where the determinants are evaluated as endomorphisms of  $E$ , and  $q, t$  are indeterminates. This may be viewed as a bi-graded character for  $S(E) \otimes \Lambda(E)$ , the tensor product of the symmetric and exterior algebras of  $E$ .

In his 1963 paper on invariants of finite reflection groups [1], Solomon explicitly determined the structure of the  $W$ -invariants of  $S(E) \otimes \Lambda(E)$ . At the level of characters, his structure theorem implies

$$\langle 1_W, \delta_W(q, t) \rangle_W = \prod_{i=1}^{\ell} \frac{1 - qt^{d_i-1}}{1 - t^{d_i}}, \quad (1)$$

where  $d_1, \dots, d_{\ell}$  are the degrees ( $\ell = \dim E$ ),  $1_W$  denotes the trivial character of  $W$ , and  $\langle f, g \rangle_W := |W|^{-1} \sum_{w \in W} f(w)g(w)$  is the usual pairing of real-valued class functions  $f$  and  $g$ .

Henceforth, assume that  $W$  is a Weyl group with an irreducible root system  $R \subset E$  of rank  $\ell$  and simple reflections  $S = \{s_1, \dots, s_{\ell}\}$ . Note that by setting  $q = 1$  and letting  $t \rightarrow 1$  in (1), we obtain the quantity  $\nu(R)$ .

We let  $s_0 \in W$  denote the reflection corresponding to the highest root and set  $S_0 = S \cup \{s_0\}$ . One may interpret  $S_0$  as the  $W$ -image of the simple reflections of the associated affine Weyl group  $\widehat{W}$ .

Following Steinberg (see Section 3 of [2]), the action of  $\widehat{W}$  on  $E$  descends to a  $W$ -action on the  $\ell$ -torus  $E/Q$  (where  $Q$  denotes the root lattice), and the decomposition of  $E$  into simplicial alcoves by the arrangement of affine hyperplanes associated to  $R$  induces a simplicial decomposition of  $E/Q$  with a compatible  $W$ -action. Moreover, the  $W$ -stabilizers of the faces of  $E/Q$  are (up to conjugacy) generated by the various proper subsets of  $S_0$ .

Given  $w \in W$ , Steinberg computes the Euler characteristic of the  $w$ -fixed subcomplex of  $E/Q$  in two different ways (see Theorem 3.12 of [2]), thereby obtaining the identity

$$\det(1 - w) = \sum_{J \subset S_0} (-1)^{|S| - |J|} 1_{W_J}^W(w), \quad (2)$$

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<sup>1</sup>Work supported by NSF grant DMS-0245385.

where  $W_J$  denotes the reflection subgroup generated by  $J$ , and  $1_{W_J}^W$  denotes the permutation character of the action of  $W$  on  $W/W_J$ . It is important to note that  $J$  ranges over *proper* subsets of  $S_0$ .

Steinberg actually proves a more general identity that involves twisting by an involution; the above instance corresponds to the trivial involution. One may also recognize (2) as a companion to the more familiar identity

$$\det(w) = \sum_{J \subseteq S} (-1)^{|J|} 1_{W_J}^W(w).$$

Now consider the evaluation of

$$\lim_{t \rightarrow 1} \langle \delta_W(1, 0), \delta_W(1, t) \rangle_W.$$

First, notice that  $\delta_W(1, t) \rightarrow 1_W$  as  $t \rightarrow 1$ , so we obtain

$$\lim_{t \rightarrow 1} \langle \delta_W(1, 0), \delta_W(1, t) \rangle_W = \langle \delta_W(1, 0), 1_W \rangle_W = 1 \quad (3)$$

by setting  $(q, t) = (1, 0)$  in (1).

Second, notice that  $\delta_W(1, 0)(w) = \det(1 - w)$ , so (2) implies

$$\begin{aligned} \langle \delta_W(1, 0), \delta_W(1, t) \rangle_W &= \sum_{J \subset S_0} (-1)^{|S| - |J|} \langle 1_{W_J}^W, \delta_W(1, t) \rangle_W \\ &= \sum_{J \subset S_0} (-1)^{|S| - |J|} \langle 1_{W_J}, \delta_{W_J}(1, t) \rangle_{W_J}, \end{aligned} \quad (4)$$

by Frobenius reciprocity. We can evaluate each of these terms by applying Solomon's formula to the reflection group  $W_J$ . But we need to be careful, because the action of  $W_J$  on  $E$  will have linear invariants if the rank of  $W_J$  is less than  $\ell = |S|$ . In such cases, this means that some of the degrees of  $W_J$  will equal 1, which introduces factors of  $(1 - q)/(1 - t)$  in (1). Since we have set  $q = 1$ , these factors vanish.

Thus (4) should be restricted to  $\ell$ -subsets of  $S_0$ , and we obtain

$$\langle \delta_W(1, 0), \delta_W(1, t) \rangle_W = \sum_{j=0}^{\ell} \prod_{i=1}^{\ell} \frac{1 - t^{d_i^{(j)} - 1}}{1 - t^{d_i^{(j)}}},$$

where  $d_1^{(j)}, \dots, d_\ell^{(j)}$  are the degrees of  $W_J$  for  $J = S_0 - \{s_j\}$ . Comparing this with (3) in the limit  $t \rightarrow 1$ , we obtain the "curious identity"

$$\sum_{j=0}^{\ell} \prod_{i=1}^{\ell} \frac{d_i^{(j)} - 1}{d_i^{(j)}} = 1.$$

## REFERENCES

- [1] L. Solomon, *Invariants of finite reflection groups*, Nagoya Math. J. 22 (1963) 57–64.
- [2] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. (1968), no. 80.

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